Stochastic Calculus and Mathematical Finance

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Syllabus

Discrete time finance (3 lectures, 1.5 hours each)

We discuss basic probability including conditional expectations and martingales in the Binomial tree setting. We then cover options pricing using no arbitrage principle, equivalent martingale measure, etc. in the simpler discrete setting. We may later cover pricing American derivative securities

Refresher in probability theory (6 lectures)

Probability measure, random variable, PDF, CDF, CF – Conditional prob. Law of large numbers, central limit theorem

We will discuss elementary probability ideas as above and also illustrate them in **measure theoretic setting**, so participants get to see sigma algebras, measurable functions, Reimann and Lebesgue integration (All this helps with later material). We will also discuss various modes of convergence (almost sure, in probability and in distribution) and illustrate their utility by proving law of large numbers, and central limit theorem. Related useful intermediate ideas such as characteristic functions will also be covered

Syllabus

Variance/Covariance – Independence – (Conditional) Expectation, Martingales

We will discuss conditional expectations in measure theoretic sense. This make appreciating martingales easier, central to math finance

Normal Distribution, Normal vector, Lognormal
We will do multi-variate Gaussian distributions along with other distribution families relevant to finance

Stochastic Ito calculus (10 lectures)

Brownian motion and general stochastic processes - properties

We introduce Brownian motion, its properties, construction, reflection principle, Brownian martingales. Stochastic processes, filterations

Ito Calculus: We define Ito's integral, its construction, discuss quadratic variation, Ito's formula, Multi-variate stochastic calculus. Integration with respect to semi-martingales. Feynman-Kac formula, Kolmogorov forward equations, Levy's characterization of Brownian motion, Martingale representation theorem, Girsanov's Theorem, We cover stochastic differential equations, apply it many financial settings – interest rate models, Vasiczek, Hull and White. Relation of SDE's to PDE's. Multi-dimensional Feyman-Kac. Review of Stochastic Calculus

Syllabus

Risk neutral derivatives pricing plus numerics (7 lectures)

We study no arbitrage pricing theory, derive Black Scholes, study equivalent risk neutral pricing measure, Delta Hedging, forwards and futures, Dividend paying stocks.
Pricing interest rate derivatives

Monte Carlo methods in finance

We cover Monte Carlo methods for options pricing including Basics of Monte Carlo, output analysis, variance reduction techniques, pricing American derivative securities.

Reference material

- Stochastic Calculus for Finance, Volume 1 and 2. Steven Shreve
- Stochastic Calculus and Financial Applications, Michael Steele
- Dynamic Asset Pricing Theory. Darrell Duffie
- Books by John C Hull …

Introduction to Derivatives

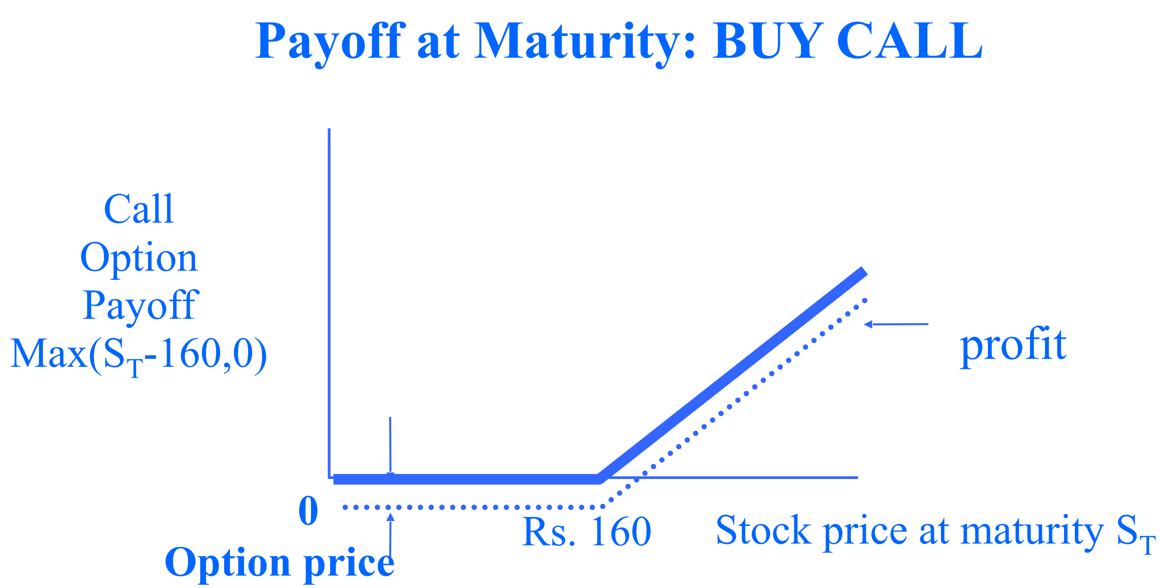
Call option

- European call option
  - An option, *not an obligation*, to purchase an underlying asset at a specified time T (expiration or maturity date) for a specified price K (strike price)
    - Example: Tata Steel stock price is Rs 158.3.
    - Option maturity October 28, 2025, strike price 160.
    - The call price is Rs. 6.55

- American call option
  - An option to purchase an underlying asset at any time *up to* a specified time T for a specified price K

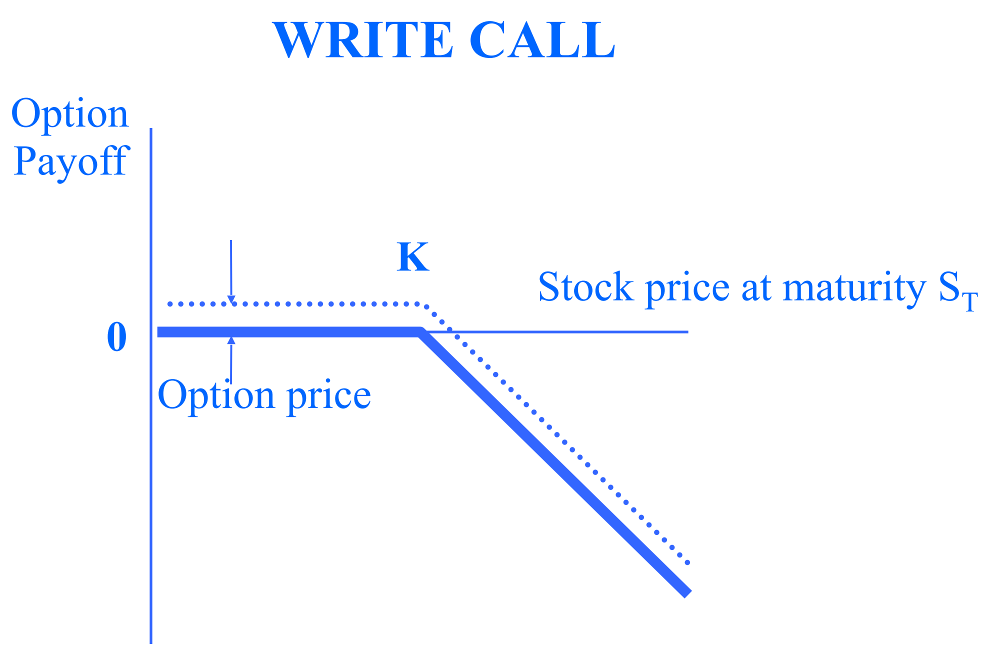
Payoff at maturity for buyer of call option

- Example: Option to purchase Tata Steel stock at Rs. 160 in October 2025
  - Option payoff if option exercised when stock price is Rs. 200
    = Max (200-160,0) = 40



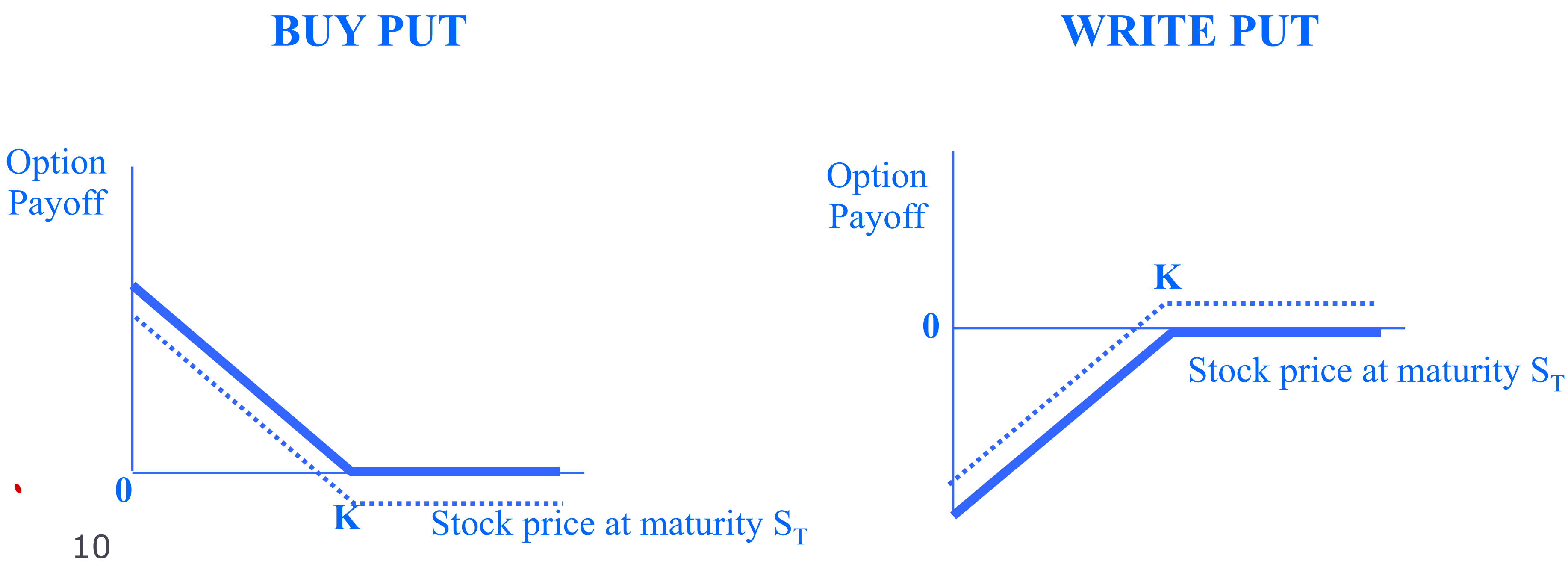
Seller or writer of option has opposite payoff

- Payoff: - Max (S_T - K, 0)
- Seller wins if the stock price remains low



European put option

- An option to sell an underlying asset at a specified time for a specified price K
  - Buyer's Payoff = Max (K-S_T, 0)
  - Insurance against falling prices



Examples of more sophisticated options

- Asian call option
  - Asset price is observed daily for T days
  - Payoff = Max (Average asset price – strike price, 0)

- Basket put option: Multiple assets involved
  - At the maturity date average of all the assets computed
  - Payoff = Max (Strike price- average assets value, 0)

  - Example: Strike price may be Rs. 500 and the basket may comprise shares of Infosys, Tata Steel, Reliance Industries.
  - This option has a payoff if exercised when the average of the three share prices is below Rs. 500.

Examples of Options on Multiple Assets

- Basket call option

$$([c_1 S_1(T) + c_2 S_2(T) + \ldots + c_d S_d(T)] - K)^+$$

- Out-performance call option

$$(\max\{c_1 S_1(T), c_2 S_2(T), \ldots, c_d S_d(T)\} - K)^+$$

- Barrier put option

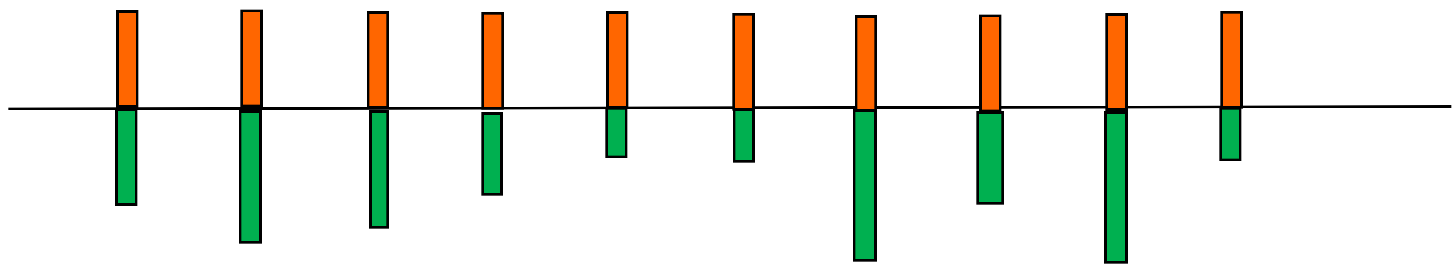
$$I(\min_{i=1,..,n}\{S_2(t_i) < b\}(K - S_1(T))^+$$

- Quantos $S_2(T)(S_1(T) - K)^+$

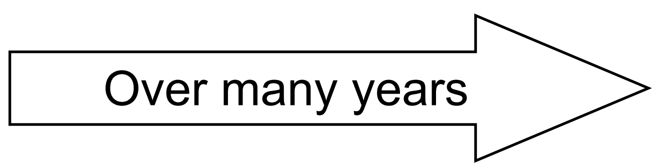
They all have an associated American version

Interest Rate Swaps, Swaptions

Fixed payment leg: Example 6% of notional amount



Floating payment leg: Example SOFR + 0.5%
(Secured Overnight Financing Rate)

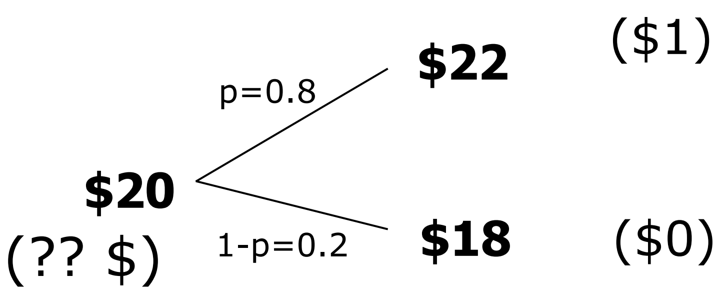


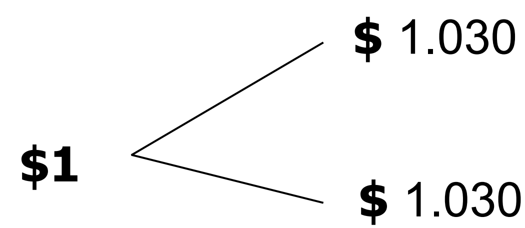
Market size $ 580 trillion 2024

No Arbitrage Principle to Price Options

Simplistan: One period Binomial model

- Stock price at $20
- After three months it takes two values $22 and $18
- What is value of European call option with strike price $21 that matures in three months?
- Risk free rate of return is 12% per annum



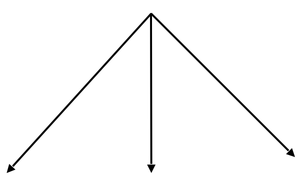


Should the option price be (1 × 0.8+0 × 0.2) /1.03 = $0.78?

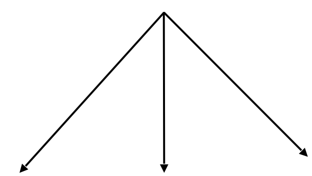
No arbitrage principle

Portfolio 1: 2 kg ketchup bottle

Portfolio 2: 4 half kg ketchup bottles



Economy scenarios 1 year later

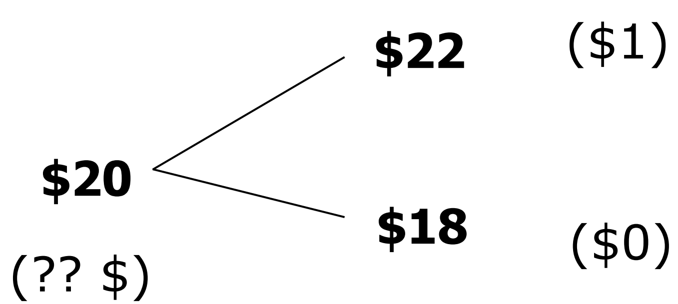


value	\$ 10	\$ 9	\$ 8	\$ 10	\$ 9	\$ 8

Two portfolio of securities having same value in all scenarios of the world should have the same price
Otherwise, by buying low and selling high we have `arbitrage'

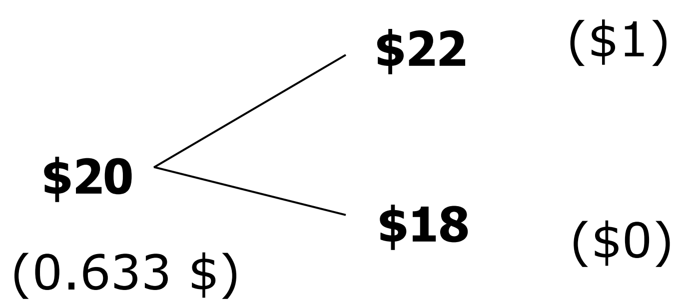
Creating a replicating portfolio

- Purchase x stocks and invest $ y in risk free security
  - Scenario 1 wealth equals 22 x + 1.030 y.
    - Set it equal to $1
  - Scenario 2 wealth equals 18 x + 1.030 y
    - Set it equal to $0
  - Then x = 1/4 and y = - $ 4.367 provides a replicating portfolio



Option price under no arbitrage principle

- Purchasing ¼ of stock and borrowing \$4.367 gives the same payoff as the option.
- The price of the two portfolios must be same
- Price of option = $\frac{1}{4} \times 20 - 4.367 =$ \$ 0.633



Arbitrage scenarios

- Suppose the price of call option was $0.75.
  - Then buy low and sell high
  - Buy ¼ stock for $5, sell a call for $0.75 and borrow $4.367 at risk free rate. Surplus of $0.117

$22 Portfolio value = 22/4 −4.367 × 1.030 -1
= 0

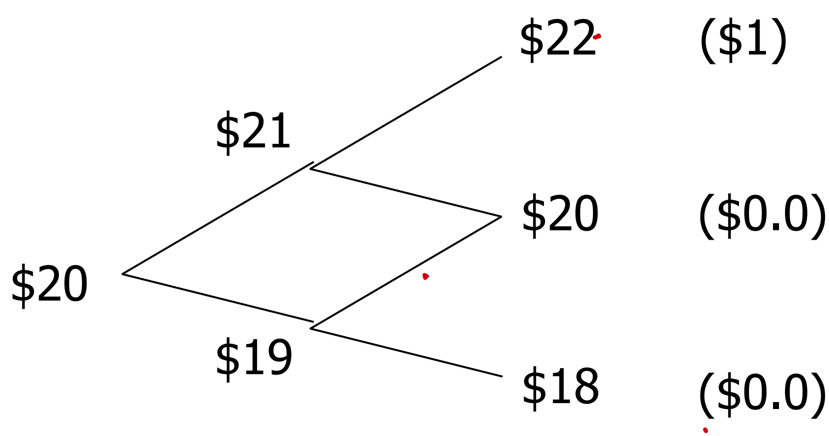
$20

$18 Portfolio value = 18/4 −4.367 × 1.030 -0
= 0

Arbitrage as sure wealth at no risk!
Similarly, when option price < $ 0.633
No arbitrage principle implies price =$0.633

Two period example

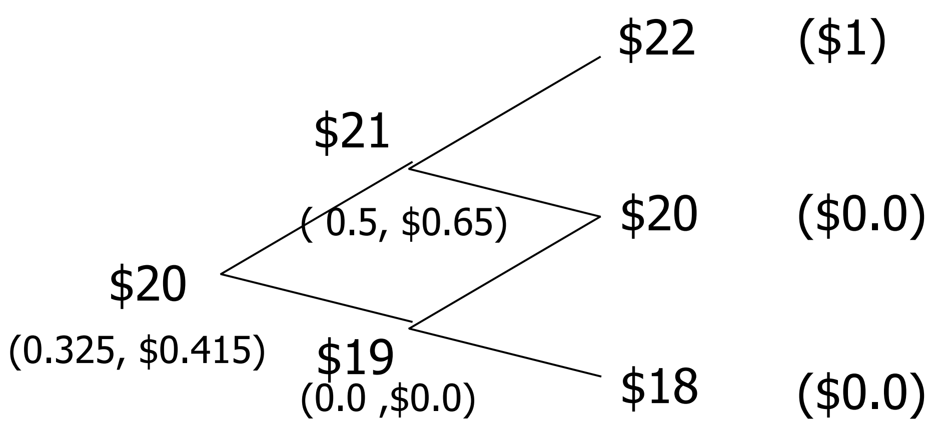
Again we price a call option with strike $21 using a two period model. Interest rate is 12% per annum and each time step is 1.5 months long



What is the no-arbitrage price for the call option?

Two period example: Solution

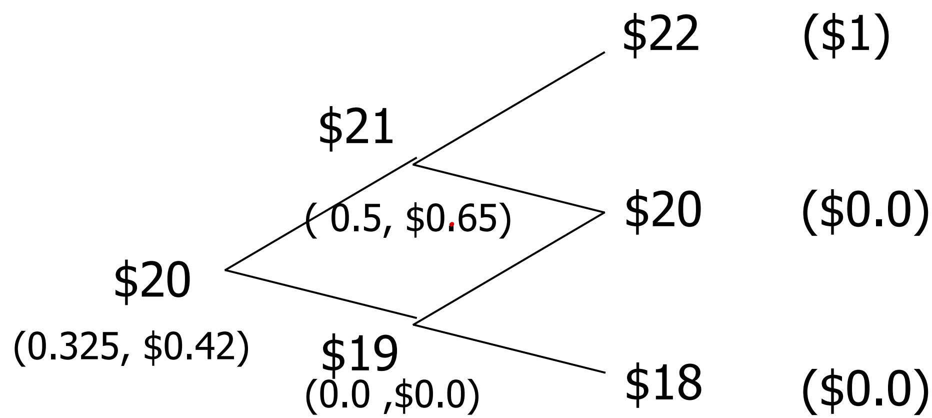
Call option with strike $21. Interest rate is 12% per annum and each time step is 1.5 months long



We can replicate the payoff from the option along each path

Two period example: Self ~~Replicating~~ Portfolio

financing

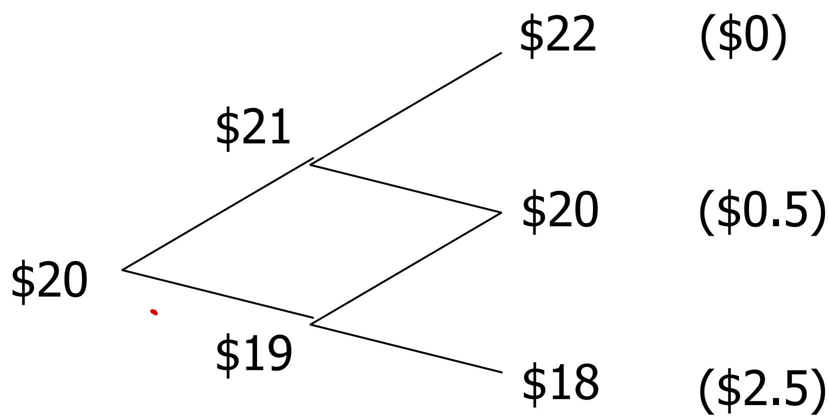


The self ~~replicating~~ portfolio :

- Start with $0.415, borrow $6.08 and purchase 0.325 stock at $6.5
- If stock goes up, the portfolio value at time 1 equals $0.65. Increase the borrowed amount from $6.175 to $9.85 and use that to change stock holding from 0.325 to 0.5
- If the stock goes down, your portfolio is worth zero…log out and spend time with your family

Class exercise 1: Price the put below

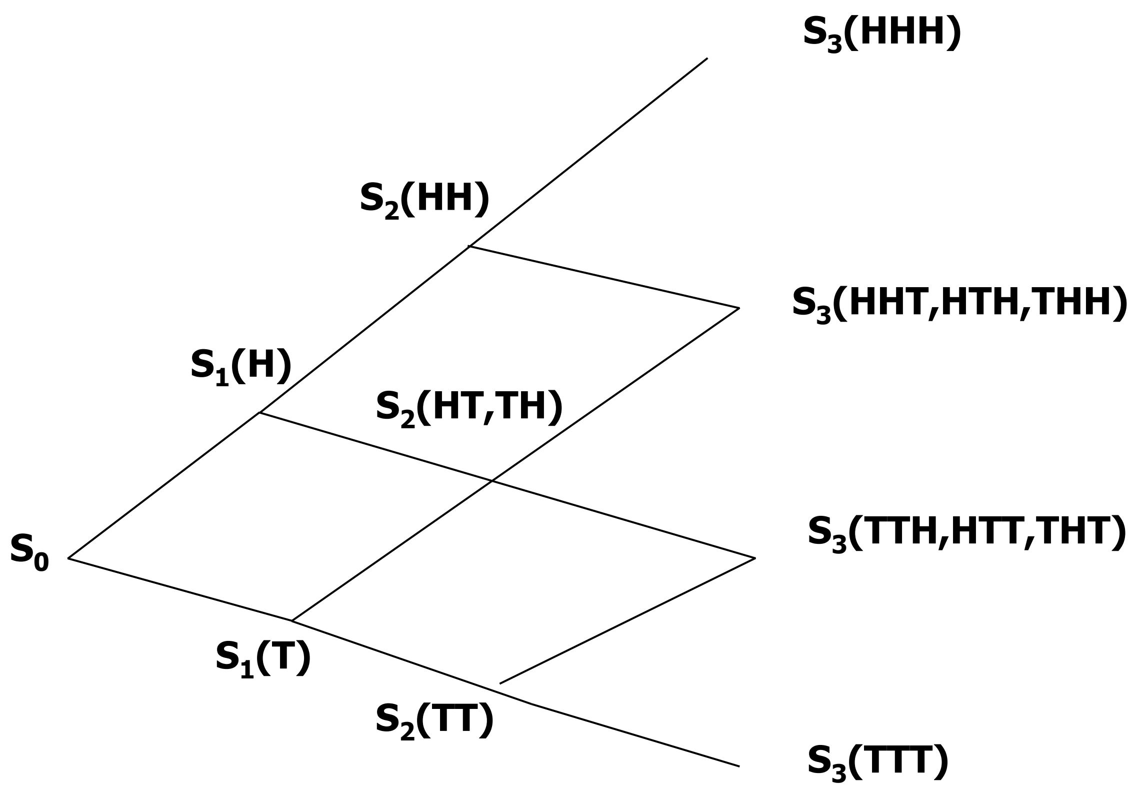
Price the European put with strike $20.5 using a two period model. Interest rate is 12% per annum and each time step is 1.5 months long



What is the no-arbitrage price for the put option?

Multi-Period Binomial Model, Path Dependent Options

- The analysis extends to multiple periods even to price options with path dependent payoffs.

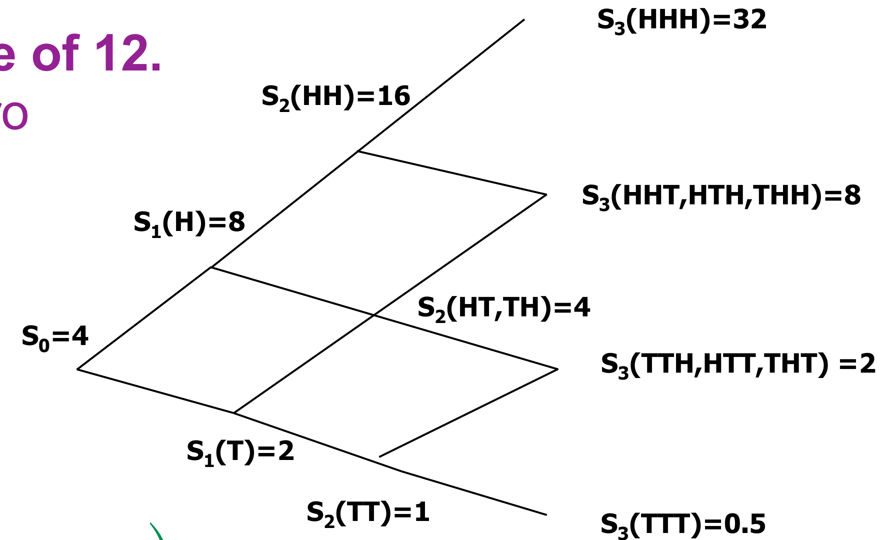


A Numerical Example: Pricing an Asian Option

Price at each step either doubles or halves.
3 time periods. Interest of 25% at each step.

$S_0 = 4$

Price an Asian option with strike price of 12.
Payoff is average of 3 periods -12 or zero whichever is higher



$$V_3(HHH) = max\left(\frac{1}{3}(S_1(H) + S_2(HH) + S_3(HHH)) - 12{,}0\right) = \sout{8/3}\ 20/3$$

$$V_3(HHT) = max\left(\frac{1}{3}(S_1(H) + S_2(HH) + S_3(HHT)) - 12{,}0\right) = 0$$

Can check that $V_2(HH) = \sout{8/9}\ 8/3$

Pricing through risk neutral probabilities: Lil' bit of mathematical magic!

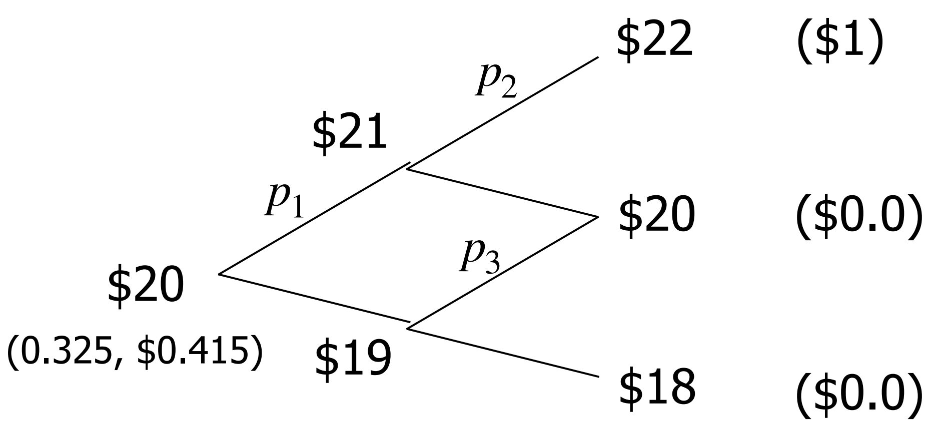
- Risk neutral probabilities: Probabilities with which asset has the same rate of return as risk free asset.
- $22\,p + 18\,(1-p) = 20 \times 1.030$

  $p = 0.6523$
- Discounted expected payoff from option price under risk neutral probabilities
  - One period model
    - $= 1/1.030 \times (1\,p + 0\,(1-p)) =$ **\$0.633 (call option)**
    - $= 1/1.030 \times (0\,p + 2\,(1-p)) =$ \$0.675 (put option)

Pricing through risk neutral probabilities:
Lil' bit of mathematical magic!

- Two period model $p_1 = 0.65,\ p_2 = 0.6575$

$$\frac{1}{1.015^2}\left(p_1 p_2 \times 1 + p_1(1-p_2)\times 0 + (1-p_1)p_3 \times 0 + (1-p_1)(1-p_3)\times 0\right) = 0.415$$



No arbitrage condition guarantees risk neutral probabilities

- No arbitrage condition implies that $d < 1 + r < u$
- Then, risk neutral probability p exists
  - $p\,u + (1-p)\,d = (1+r)$
  - $p = (1+r-d)/(u-d)$

The BIG idea

- Under risk neutral probabilities both the risky asset and the riskless asset grow on average at rate 1+r

- Therefore any self-~~replicating~~ financing portfolio grows at rate 1+r

- Discounted value of such a option replicating portfolio does not change on average `it is a Martingale'

- Value of any option whose payoff can be replicated by a self-~~replicating~~ portfolio equals the expectation of its discounted payoff under the risk neutral probabilities.

Basic Probability Notes

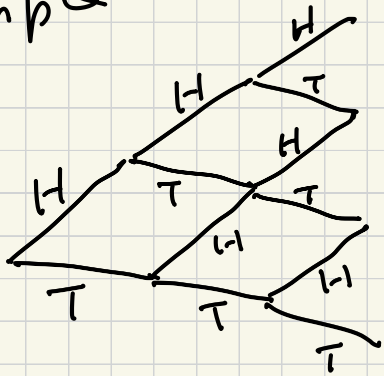
Consider sample space

Ω_N containing 2^N

outcomes $\omega = (\omega_1, \ldots, \omega_N)$

each $\omega_i \in \{H, T\}$

Example



$|\Omega_3| = 8$

On sample space

we have probability

P. $P(\omega) \geq 0$

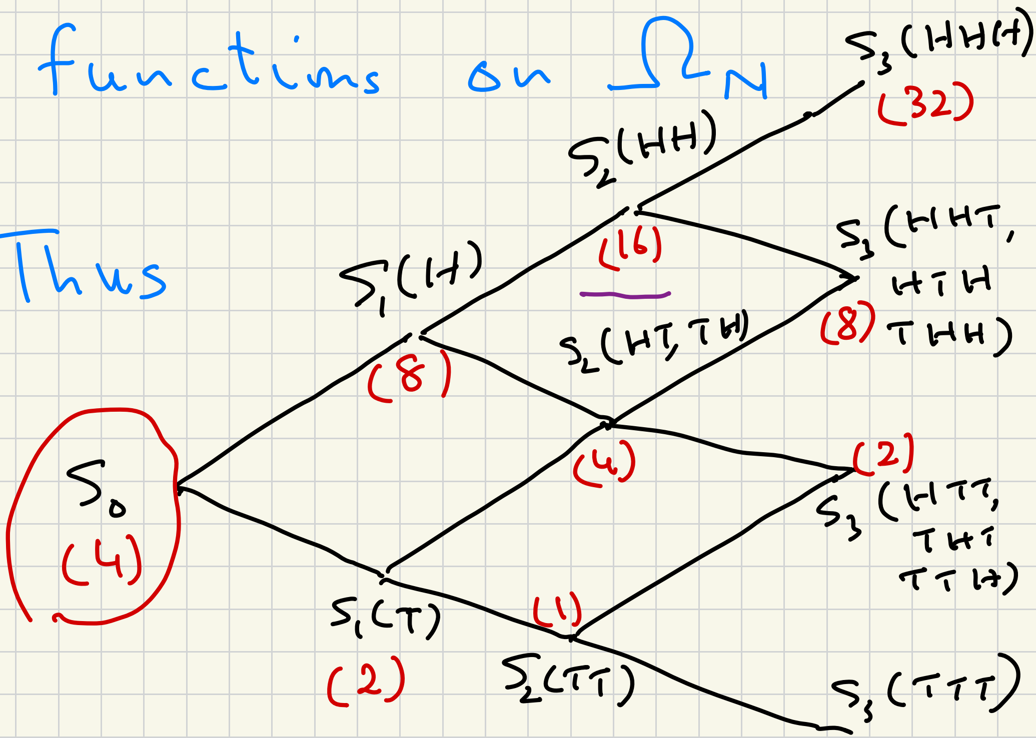
& $P(A) = \sum_{\omega \in A} P(\omega)$

$P(\Omega_N) = 1$.

Random variables are

functions on Ω_N

Thus



Here, r.v. S_0 constant on all ω.

S_1 takes two values

S_2 takes three values

S_3 takes four values

Expectation

$$EX = \sum_{\omega \in \Omega} X(\omega) P(\omega)$$

Suppose

$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
$P(\omega)$	0.1	0.2	0.15	0.05	0.05	0.2	0.1	0.15

What is ES_2 ?

$$16(.1+.2)+4(.15+.05+.05+.05)$$
$$+1(0.1+0.15)$$

Conditional Expectation

Recall that

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

and

$$E(X \mid Y=y) = \sum_x x P(X=x \mid Y=y)$$

We now define

$$P(\omega \mid (\omega_1, \ldots \omega_n)) = \frac{P(\omega)}{P(\omega_1, \ldots, \omega_n)}$$

&

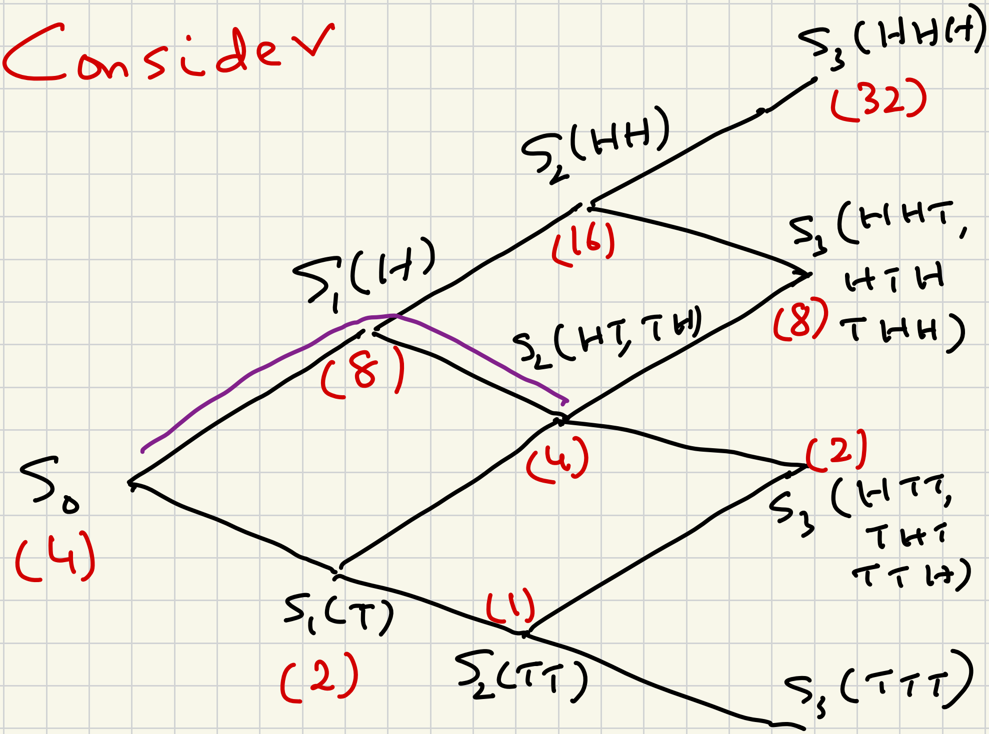
$$E_n[X \mid (\omega_1, \ldots \omega_n)] = \sum_{\forall (\omega_{n+1}, \ldots, \omega_N)} X(\omega) \frac{P(\omega)}{P(\omega_1, \ldots, \omega_n)}$$

When it is obvious we write

$$E_n[X \mid (\omega_1, \ldots, \omega_n)]$$

as $E_n X$

Consider



$$E_2\left[S_3 \,|\, (H, T)\right]$$

$$= S_3(HTH)\frac{P(HTH)}{P(HT)}$$

$$+ \; S_3(HTT)\frac{P(HTT)}{P(HT)}$$

Usually $P(HTT) = p(1-p)^2$

$P(HT) = p(1-p)$ & so on

because coin flip are independent. Then

$$E_2[S_3 \mid (H,T)]$$

$$= S_3(HTH)\,p$$

$$+ S_3(HTT)(1-p).$$

$E(S|X)$

is a random variable

that is a fn. of

X, call it $f(x)$.

Then

$$f(x) = E(S|X=x)$$

Can show

$$E\left[E(S|X,Y)|X\right]$$
$$= E[S|X].$$

A stochastic process or a collection of random variables $(X_1, X_2, \ldots, X_N)$ is called a martingale if

$$E\left(X_n \mid (X_1, \ldots, X_{n-1})\right) = X_{n-1}$$

On average it is not changing.

<u>Example</u> $(Y_1, Y_2, \ldots Y_N)$ are independent with mean zero.

$$S_n = \sum_{i=1}^{n} Y_i$$

$$\underline{E(S_{n+1} \mid (Y_1, \ldots, Y_n))}$$

$$= S_n + E(Y_{n+1} \mid (Y_1, \ldots, Y_n))$$

$$= S_n$$

& hence $(S_n : 0 \le n \le N)$ is a martingale.

Similarly, if each

$$E Y_i = 1$$

then for $M_n = \prod_{i=1}^{n} Y_i$

$$E[M_{n+1} \mid (Y_1, \ldots, Y_n)]$$

$$= M_n \, E[Y_{n+1} \mid (Y_1, \ldots, Y_n)]$$

$= M_n$

and hence is a martingale.

Suppose $(X_1, \ldots, X_N)$ is a mgle

$$E(X_{n+1} \mid (X_1, \ldots, X_n)) = X_n.$$

Then

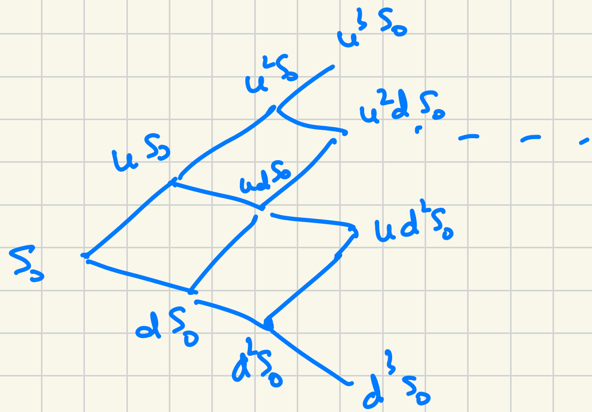
$$E\left[E X_{n+2} \mid (X_1, \ldots, X_{n+1}) \mid (X_1, \ldots, X_n) \right]$$

$$= X_n$$

$$\Rightarrow E X_{n+2} \mid (X_1, \ldots, X_n) = X_n$$

$$\Rightarrow \quad E\left[X_m \mid (X_1, \dots, X_n)\right] = X_n \quad \text{for } m > n.$$

Consider the stock price model



$$S_n = u^{\#H} d^{\#T} S_0$$

#H : no. of heads in n trials.

No arbitrage

$\Rightarrow$ $$d < 1+r < u.$$

Easy to check

with $$\tilde{p} = \frac{1+r-d}{u-d}$$

$$\tilde{q} = 1 - \tilde{p}$$

we have

$$\tilde{E}_n \left[S_{n+1} \,\middle|\, (\omega_1, \dots, \omega_n) \right]$$

$$= u S_n \tilde{p} + d S_n \tilde{q}$$

$$= S_n \left[\frac{u(1+r-d)}{u-d} + d \frac{(u-(1+r))}{u-d} \right]$$

$$= S_n (1+r)$$

$$\Rightarrow \tilde{E}_n \left[\frac{S_{n+1}}{(1+r)^{n+1}} \,\middle|\, (\omega_1 \dots \omega_n) \right] = \frac{S_n}{(1+r)^n}$$

$\forall \ (\omega_1, \ldots, \omega_n)$.

$\Rightarrow \quad \frac{S_n}{(1+r)^n}$ is a mgle.

Consider a portfolio that has X_n wealth at $(\omega_1, \ldots, \omega_n)$.

It purchases δ_n amount of stock & invests rest in risk free rate

$$X_{n+1} = \delta_n S_{n+1} + (1+r)(X_n - \delta_n S_n)$$

$\tilde{E}_n[X_{n+1}]$ (hiding $(\omega_1, \ldots, \omega_n)$)

$$= \delta_n \tilde{E}_n S_{n+1} + (1+r)(X_n - \delta_n S_n)$$

$$= (1+r) X_n$$

$\Rightarrow$ Any discounted self financing portfolio process

$$\left(\frac{X_n}{(1+r)^n} : n \geq 0 \right)$$

is a mgle.

$\Rightarrow$ Since options can be exactly replicated by a self financing portfolio process

⇒ Discounted option price process

$$\left(\frac{V_n}{(1+r)^n} : 0 \le n \le N \right)$$ is

a martingale.

Then $\widetilde{E}_n \frac{V_{n+1}}{(1+r)^{n+1}} = \frac{V_n}{(1+r)^n}$

Can be seen that

$$\Longrightarrow \quad \widetilde{E}_n \frac{V_m}{(1+r)^m} = \frac{V_n}{(1+r)^n} \quad m \ge n.$$

$$\Longrightarrow \quad V_0 = \widetilde{E} \frac{V_N}{(1+r)^N}$$

But V_N – option pay-off function. CALL IT G_N

e.g. $G_N = (S_N - K)^+$

for call option

$$G_N = \max_{0 \le n \le N} S_n - S_N$$

'Lookback option'

$$G_N = I\left(\min_n S_n > B\right)(K - S_N)^+$$

'Down & out barrier option.'

& V_0 is option price

$\Rightarrow$ Option price is

$$\tilde{E}\left((1+r)^{-N} G_N\right)$$

i.e.,

Expected discounted payoff fn. under risk neutral probabilities

Binomial Tree Model is Complete

- Every security V_N can be hedged using a replicating portfolio and hence has a unique price.

- If the tree was trinomial, and there were two securities as before
  - not every security could be replicated (incomplete market),
  - only bounds could be developed on prices using the no-arbitrage condition

Fundamental Theorem in Option Pricing

The no arbitrage condition implies existence of equivalent martingale probability $\tilde{P}$ under which discounted asset prices

$$\left(\frac{S_n}{(1+r)^n} : 0 \leq n \leq N \right)$$

and hence attainable option price process

$$\left(\frac{V_n}{(1+r)^n} : 0 \leq n \leq N \right)$$

are martingales

Fundamental Theorem in Option Pricing

If there exist equivalent martingale probability measure $\tilde{P}$, arbitrage is no longer feasible

Arbitrage: Consider the portfolio process $(X_1, X_2, \ldots, X_N)$

$X_0 = 0$ and $X_N \geq 0$ for all outcomes

and $X_N > 0$ with positive probability

Then, $\tilde{E}(X_N) > 0$. But existence of $\tilde{P}$ implies $\tilde{E}(X_N) = X_0 = 0$

A contradiction!